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$*^s$-Modules

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Abstract

In this paper we introduce the notion of $*^s$-modules (s denotes *static*) as a generalization of $*$-modules different from $*^n$-modules. The class of $*^s$-modules contains also the class of self-small abelian groups faithfully flat over their endomorphism rings. We study characterizations of $*^s$-modules and extend successfully some results in the theory of $*$-modules. The relations between $*^s$-modules and $*^n$-modules are also considered. Finally we give characterizations of $*^s$-modules with some special properties.
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Introduction

The theory of equivalences between module subcategories, originates in the well-known theory of Morita equivalence, had been studied extensively, see [4,8,11,14,19] etc. Modules playing important roles in the theory are the progenerators, quasi-progenerators, tilting modules, $*$-modules and others. In particular, progenerators are just $*$-modules which are generators; quasi-progenerators are just $*$-modules which are Σ-self-generators while tilting modules (of projective dimension $\leqslant 1$) are just $*$-modules which generate all the injective modules.

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Recently, the author and the others studied $*^n$-modules as a generalization of both $*$-modules and tilting modules of projective dimension $\leqslant n$ in sense of [12,17,18]. In fact, $*$-modules are just $*^1$-modules while tilting modules of projective dimension $\leqslant n$ are just $*^n$-modules P such that the subcategory $\mathrm{Pres}^n(P)$ contains all the injective modules and P admits a finitely generated projective resolution.

In this note, we will present another generalization of $*$-modules which is different from $*^n$-modules.

Namely, we introduce the notion of $*^s$-modules, where s denotes *static*, by replacing the subcategory $\mathrm{Gen}(P)$ in the theory of $*$-modules with the subcategory $\mathrm{Stat}(P)$. Some results on $*$-modules are successfully extended to our settings (cf. Propositions 2.2, 4.6 and 5.5 etc.). We also compare $*^s$-modules with $*^n$-modules. In fact, we show that a $*^s$-module P is a $*^n$-module if and only if $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$ while a $*^n$-module P is a $*^s$-module if and only if $\mathrm{Stat}(P) \subseteq \mathrm{Pres}^n(P)$ (Propositions 3.1 and 3.2). In particular, all $*^2$-modules are $*^s$-modules. Interesting, examples of $*^s$-modules also arise from the theory of abelian groups. Let P be a selfsmall abelian group which is faithfully flat over its endomorphism ring, then P is a $*^s$-module (Proposition 3.6). We refer to the papers [1,2,10], etc., for the study of such abelian groups. Main characterizations of $*^s$-modules are given in Section 4. In this section we show that an R-module P with $A = \mathrm{End}_R P$ is a $*^s$-module if and only if ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$ and $\mathrm{Ker}\, T_P^{i\geqslant 0} = 0$ (Theorem 4.4). $*^s$-Modules with some special properties are considered in Section 5. For example, selfsmall R-modules which are faithfully flat over their endomorphism rings are characterized as $*^s$-modules flat over their endomorphism rings (Proposition 5.1). We also show the following result (Theorem 5.2). Let P be an R-module. Then P is a $*^s$-module such that $\mathrm{Stat}(P)$ is a coresolving subcategory, if and only if P is selfsmall and $\mathrm{Stat}(P) = P^{\perp}$, if and only if $\mathcal{I} \subseteq \mathrm{Stat}(P) \subseteq P^{\perp}$ and ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$.

1. Preliminaries

Throughout this paper, all rings will be associative with non-zero identity and modules will be left modules. For a ring R, R-Mod (Mod-R) denotes the category of all left (right) R-modules. By a subcategory, we mean a full subcategory closed under isomorphisms.

From now on, we fix R a ring and P an R-module with $A = \mathrm{End}_R\, P$. Note that P is also a right A-module. Denote that $H_P = \mathrm{Hom}_R(P, -)$ and $T_P = P \otimes_A -$.

We denote by $P^{\perp} = \{M \in R\text{-Mod} \mid \mathrm{Ext}_R^i(P, M) = 0 \text{ for all } i \geqslant 1\}$ and $\mathrm{Ker}\, T_P^{i\geqslant n} = \{M \in A\text{-Mod} \mid \mathrm{Tor}_i^A(P, M) = 0 \text{ for all } i \geqslant n\}$ for a fixed $n \geqslant 0$ (here we assume that $\mathrm{Tor}_0^A(P, -) = T_P$). Also we denote by $\mathrm{Ad}\, P$ the class of modules isomorphic to direct sums of copies of the R-module P and by $\mathrm{Add}\, P$ the class of modules isomorphic to direct summands of modules in $\mathrm{Ad}\, P$. We denote by ${}_A\mathcal{P}$ the class of all projective A-modules.

An R-module M is n-presented by P (or has P-codominant dimension $\geqslant n$), for some $n \geqslant 1$, if there exists an exact sequence $P_n \to \cdots \to P_2 \to P_1 \to M \to 0$ with $P_i \in \mathrm{Add}\, P$ for each i. We denote by $\mathrm{Pres}^n(P)$ the category of all modules with P-codominant dimension $\geqslant n$. Of course, for every n, we have that $\mathrm{Pres}^{n+1}(P) \subseteq \mathrm{Pres}^n(P)$. Note that $\mathrm{Pres}^2(P)$ and $\mathrm{Pres}^1(P)$ are just the familiar subcategories $\mathrm{Pres}(P)$ and $\mathrm{Gen}(P)$, respectively.

P is said to be (n,t)-quasi-projective (here we assume that $n \geqslant t \geqslant 1$) if, for any exact sequence $0 \to M \to P_t \to \cdots \to P_1 \to N \to 0$ with $M \in \mathrm{Pres}^{n-t}(P)$ and $P_i \in \mathrm{Add}\, P$ for each i, the induced sequence $0 \to H_P M \to H_P P_t \to \cdots \to H_P P_1 \to H_P N \to 0$ is exact [17]. Note that the notions of $(1,1)$-quasi-projective, $(2,1)$-quasi-projective, $(2,2)$-quasi-projective and $(n,1)$-quasi-projective respectively are just the notions of Σ-quasi-projective [8,13], w-Σ-quasi-projective [5], semi-Σ-quasi-projective [13] and n-quasi-projective [18], respectively.

P is said to be selfsmall if, for any set X, there is the canonical isomorphism

$$\mathrm{Hom}_R\big(P, P^{(X)}\big) \simeq \mathrm{Hom}_R(P,P)^{(X)}.$$

It is well known that (T_P, H_P) is a pair of adjoint functors and there are the following canonical homomorphisms for any R-module M and any A-module N:

$$\rho_M : T_P H_P M \to M \quad \text{by} \quad p \otimes f \to f(p);$$
$$\sigma_N : N \to H_P T_P N \quad \text{by} \quad n \to [p \to p \otimes n].$$

We denote by $\mathrm{Stat}(P)$ the class of P-static modules, i.e., the R-modules M such that ρ_M is an isomorphism. We also denote by $\mathrm{Costat}(P)$ the class of P-costatic modules, i.e., the A-modules N such that σ_N is an isomorphism. Note that the pair of functors (H_P, T_P) defines a basic equivalence between $\mathrm{Stat}(P)$ and $\mathrm{Costat}(P)$. It is also well known that $\mathrm{Stat}(P) \subseteq \mathrm{Pres}(P)$ for any R-module P.

A subcategory is coresolving if it contains all injective objects and is closed under extensions and cokernels of monomorphisms. Dually, a subcategory is resolving if it contains all projective objects and is closed under extensions and kernels of epimorphisms.

Finally, we recall the following characterizations of $*$-modules. For more results on the theory of $*$-modules, we refer to the papers [5–7,9,11,15,16], etc.

Theorem 1.1 [5]. *Let P be an R-module with $A = \mathrm{End}_R\, P$. Then the following statements are equivalent.*

(1) *P is a $*$-module.*
(2) *P is selfsmall, w-Σ-quasi-projective, and* $\mathrm{Gen}(P) = \mathrm{Pres}(P)$.
(3) *P is selfsmall and, for any $M \leqslant P_M$ with $P_M \in \mathrm{Add}\, P$, $M \in \mathrm{Gen}(P)$ if and only if $\mathrm{Ext}^1_R(P,M) \to \mathrm{Ext}^1_R(P,P_M)$ is injective canonically.*
(4) *P is selfsmall and, for any exact sequence $0 \to L \to M \to N \to 0$ with $M, N \in \mathrm{Gen}(P)$, the induced sequence $0 \to H_P L \to H_P M \to H_P N \to 0$ is exact if and only if $L \in \mathrm{Gen}(P)$.*

2. Basic properties on $*^s$-modules

Note that, for P a $*$-module, the subcategory $\mathrm{Stat}(P)$ equals $\mathrm{Gen}(P)$ [5], so it is interesting to consider what happens if we replace the subcategory $\mathrm{Gen}(P)$ in the theory of $*$-modules with the subcategory $\mathrm{Stat}(P)$. This leads the following definition.

Definition 2.1. An R-module P is said to be a $*^s$-module provided that P is selfsmall and that any exact sequence $0 \to L \to M \to N \to 0$ with $M, N \in \mathrm{Stat}(P)$ remains exact after applying the functor H_P if and only if $L \in \mathrm{Stat}(P)$.

It is easy to see that $*^s$-modules with $\mathrm{Stat}(P) = \mathrm{Gen}(P)$ are just $*$-modules (cf. Theorem 1.1). Hence the notion of $*^s$-modules is a generalization of $*$-modules.

For a $*^s$-module P, the subcategory $\mathrm{Stat}(P)$ has the following properties.

Proposition 2.2. *Let P be a $*^s$-module. Then*

(1) *The functor H_P preserves short exact sequence s in* $\mathrm{Stat}(P)$.
(2) *For any $M \in \mathrm{Stat}(P)$, there is an infinite exact sequence $\cdots \to P_n \to \cdots \to P_1 \to M \to 0$ which remains exact after applying the functor H_P, where $P_i \in \mathrm{Add}\, P$ for each i.*
(3) *For any exact sequence $0 \to L \to M \to N \to 0$ which is also exact after applying the functor H_P, if two of its terms are in $\mathrm{Stat}(P)$, then so is the third one.*

Proof. (1) Follows from the definition of $*^s$-modules.

(2) Note that $\mathrm{Stat}(P) \subseteq \mathrm{Pres}(P)$, so, for any $M \in \mathrm{Stat}(P)$, we have an exact sequence $0 \to M_1 \to P_1 \to^{f_1} M \to 0$ with $P_1 = P^{(H_P M)} \in \mathrm{Add}\, P$ and f_1 the evaluation map. Since the sequence is clearly exact after applying the functor H_P and $P_1, M \in \mathrm{Stat}(P)$, we obtain that $M_1 \in \mathrm{Stat}(P)$. By repeating the process to M_1, and so on, we finally obtain the desired exact sequence.

(3) If $M, N \in \mathrm{Stat}(P)$, then $L \in \mathrm{Stat}(P)$ by assumptions and Definition 2.1.

Now let $L \in \mathrm{Stat}(P)$. By applying the functor $T_P H_P$ to the sequence in (3), we have the following exact commutative diagram by assumptions.

$$\begin{tikzcd}
 & T_P H_P L \arrow[r] \arrow[d, "\rho_L"] & T_P H_P M \arrow[r] \arrow[d, "\rho_M"] & T_P H_P N \arrow[r] \arrow[d, "\rho_N"] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Note that ρ_L is an isomorphism since $L \in \mathrm{Stat}(P)$. It follows easily that ρ_M is an isomorphism if and only if ρ_N is an isomorphism. Hence $M \in \mathrm{Stat}(P)$ if and only if $N \in \mathrm{Stat}(P)$. $\square$

We note that the converse of Proposition 2.2(2) holds whenever P is selfsmall [19].

The following proposition gives some properties of the subcategory $\mathrm{Costat}(P)$ for P a $*^s$-module.

Proposition 2.3. *Let P be a $*^s$-module with $A = \mathrm{End}_R\, P$. Then*

(1) ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i \geqslant 1}$. *In particular, the functor T_P preserves short exact sequences in* $\mathrm{Costat}(P)$.
(2) $\mathrm{Costat}(P)$ *is a resolving subcategory.*

Proof. (1) Since P is selfsmall, we easily obtain that ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P)$. Now for any $N \in \mathrm{Costat}(P)$, we have that $T_P N \in \mathrm{Stat}(P)$, and hence there is an infinite exact sequence $\cdots \to P_n \to \cdots \to P_1 \to T_P N \to 0$ which remains exact after applying the functor H_P, where $P_i \in \mathrm{Add}\, P$ for each i, by Proposition 2.2(2). Then, by applying the functor H_P to the sequence, we obtain an exact sequence $\cdots \to H_P P_n (\simeq A_n) \to \cdots \to H_P P_1 (\simeq A_1) \to H_P T_P N (\simeq N) \to 0$ with $A_i \in {}_A\mathcal{P}$. Since it is clearly exact after applying the functor T_P, we obtain that $\mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i \geqslant 1}$ by dimension shifting.

(2) $\mathrm{Costat}(P)$ contains all projective A-modules by (1). Now for any exact sequence $0 \to K \to M \to N \to 0$ with $N \in \mathrm{Costat}(P)$, we obtain the following exact commutative diagram by applying the functor $H_P T_P$ since $\mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i \geqslant 1}$ by (1).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_K} & & \downarrow{\scriptstyle \sigma_M} & & \downarrow{\scriptstyle \sigma_N} & & \\
0 & \longrightarrow & H_P T_P K & \longrightarrow & H_P T_P M & \longrightarrow & H_P T_P N & \longrightarrow &
\end{array}$$

Note that σ_N is an isomorphism since $N \in \mathrm{Costat}(P)$, we have that σ_K is an isomorphism if and only if σ_M is an isomorphism. That is, $K \in \mathrm{Costat}(P)$ if and only if $M \in \mathrm{Costat}(P)$. It follows that $\mathrm{Costat}(P)$ is also closed under kernels of epimorphisms and under extensions. Thus, $\mathrm{Costat}(P)$ is a resolving subcategory. $\square$

3. Relations between $*^s$-modules and $*^n$-modules

The notion of $*^n$-modules, introduced in [17,18] by the author and the others, is also a generalization of $*$-modules. It is natural to consider the relations between $*^s$-modules and $*^n$-modules. Recall that P is said to be a $*^n$-module if P is selfsmall, $(n+1)$-quasi-projective and $\mathrm{Pres}^n(P) = \mathrm{Pres}^{n+1}(P)$ [18]. Note that $*$-modules are exactly $*^1$-modules (cf. Theorem 1.1).

The following is the first result in this direction.

Proposition 3.1. *Let P be a $*^s$-module. Then P is a $*^n$-module if and only if* $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$. *In particular, P is a $*^n$-module provided that P is n-quasi-projective with $n \geqslant 2$.*

Proof. If P is a $*^n$-module, then we clearly have that $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$ by [18]. Now assume that $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$. By Proposition 2.2, we have that $\mathrm{Stat}(P) \subseteq \mathrm{Pres}^n(P)$. It follows that $\mathrm{Stat}(P) = \mathrm{Pres}^n(P)$. Hence we obtain that P is selfsmall and any exact sequence $0 \to L \to M \to N \to 0$ with $M, N \in \mathrm{Pres}^n(P)$ remains exact after applying the functor H_P if and only if $L \in \mathrm{Pres}^n(P)$ by Definition 2.1. Therefore, P is also a $*^n$-module by [18].

If P is n-quasi-projective, then $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$ by [17]. Hence P is a $*^n$-module by the first part. $\square$

Similarly, we have the following result.

Proposition 3.2. *Let P be a $*^n$-module. Then P is a $*^s$-module if and only if* $\mathrm{Stat}(P) \subseteq \mathrm{Pres}^n(P)$.

Proof. The necessity has been proved in the last proposition. Now assume that $\mathrm{Stat}(P) \subseteq \mathrm{Pres}^n(P)$, then we obtain that $\mathrm{Pres}^n(P) = \mathrm{Stat}(P)$ since $\mathrm{Pres}^n(P) \subseteq \mathrm{Stat}(P)$ due to that P is a $*^n$-module. Hence, by [18] and the definition of $*^s$-modules, we have that P is a $*^s$-module. □

A corollary of the last proposition shows that all $*^2$-modules are $*^s$-modules.

Corollary 3.3. *Let P be a $*^2$-module. Then P is a $*^s$-module.*

As to cases $n \geqslant 3$, we will see that $*^n$-modules need not be $*^s$-modules from the following example.

Example 3.4. A $*^n$-module ($n \geqslant 3$) need nor be a $*^s$-module.

Proof. In fact, let R denote the path algebra defined by the quiver $1 \to^{\alpha} 2 \to^{\beta} 3$ with the relation $\alpha\beta = 0$. Then, for this algebra, we have a projective module $P = {}^{2}_{1} \oplus {}^{3}_{2}$. Since $\mathrm{Pres}^3(P) = \mathrm{Add}\, P$, we see that P is a $*^3$-module [18]. Note that the simple module $S(3) \in \mathrm{Pres}^2(P)$ and that $S(3) \notin \mathrm{Pres}^3(P)$, so we obtain that $\mathrm{Pres}^3(P) \subsetneq \mathrm{Pres}^2(P)$ and hence P is not a $*^2$-module. Since P is projective, we have that P cannot be a $*^s$-module. Otherwise, P must be a $*^2$-module by Proposition 3.1 since projective modules are clearly 2-quasi-projective, a contradiction. □

Using the notion of $*^s$-modules, progenerators and quasi-progenerators can be characterized as follows.

Proposition 3.5. *Let P be an R-module.*

(1) *P is a progenerator if and only if P is a $*^s$-module and a generator.*
(2) *P is a quasi-progenerator if and only if P is a $*^s$-module and a Σ-self-generator.*

Proof. The necessity is clear. Now if P is a Σ-self-generator, then $\mathrm{Gen}(P) = \mathrm{Stat}(P)$. Since P is also a $*^s$-module, P is a $*$-module by Proposition 3.1. An R-module which is a $*$-module and a generator (a Σ-self-generator, respectively) is clearly a progenerator (a quasi-progenerator, respectively) [5], so the sufficient parts hold. □

It is interesting to know if there are other $*^s$-modules which are not $*^n$-modules for any n. The following result will give an answer to this question.

Proposition 3.6. *Let P be a selfsmall R-module with $A = \mathrm{End}_R\, P$. If P_A is faithfully flat, then P is a $*^s$-module.*

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence with $M, N \in \mathrm{Stat}(P)$. Assume first that the sequence is exact after applying the functor H_P, i.e., we have the induced exact sequence $0 \to H_P L \to H_P M \to H_P N \to 0$. Since P_A is flat by assumptions, after applying the functor T_P we obtain that the sequence $0 \to T_P H_P L \to T_P H_P M \to T_P H_P N \to 0$ is exact. Note that $M, N \in \mathrm{Stat}(P)$, so ρ_M and ρ_N are isomorphisms. It follows that ρ_L is an isomorphism and $L \in \mathrm{Stat}(P)$.

Now suppose that $L \in \mathrm{Stat}(P)$. After applying the functor H_P, we obtain an induced exact sequence $0 \to H_P L \to H_P M \to H_P N \to D \to 0$ for some D. Since also P_A is flat, we have that the sequence $0 \to T_P H_P L \to T_P H_P M \to T_P H_P N \to T_P D \to 0$ is exact, by again applying the functor T_P. It is easy to see that $T_P D = 0$. Since P_A is faithfully flat, we then obtain that $D = 0$ [3]. Hence $0 \to H_P L \to H_P M \to H_P N \to 0$ is exact. By Definition 2.1, we have that P is a $*^s$-module. □

We note that selfsmall abelian groups (i.e., $\mathbf{Z}$-modules) which are faithfully flat over their endomorphism rings play important roles in the studies of the theory of abelian groups (see for instance [1,2,10] etc.). The following is one example of such abelian groups. We refer to the paper [10] and the references cited there for more examples. Note that the example also shows that, in contrast to $*$-modules which are always finitely generated, $*^s$-modules are not finitely generated in general.

Example 3.7. Let P be a torsion-free abelian group of rank 2 with $\mathrm{End}\, P = \mathbf{Z}_p$. Then P is selfsmall abelian group and P is faithfully flat over its endomorphism ring $\mathbf{Z}_p$. In the case, P is a $*^s$-module which is not a $*^n$-module for any n.

Proof. P is selfsmall abelian group and P is faithfully flat over its endomorphism ring $A = \mathbf{Z}_p$, as shown in [1, Example 3.3]. To see that P is never a $*^n$-module for any n, it is sufficient to show that $\mathrm{Costat}(P) \neq \mathrm{Ker}\, T_P^{i \geqslant 1} = A\text{-Mod}$ by [18]. However, this is followed from the fact that, in the case, $\mathrm{Costat}(P)$ is not a torsion-free class by [1, Example 3.3 and Theorem 2.3]. □

4. Characterizations of $*^s$-modules

Suggested by Proposition 3.6, we have the following more general result.

Proposition 4.1. *Let P be an R-module with $A = \mathrm{End}_R\, P$. Assume that ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i \geqslant 1}$ and $\mathrm{Ker}\, T_P^{i \geqslant 0} = 0$. Then P is a $*^s$-module.*

Proof. It is easy to see that ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P)$ implies that P is selfsmall.

Now let $0 \to L \to M \to N \to 0$ be an exact sequence with $M, N \in \mathrm{Stat}(P)$. Assume first that the sequence is exact after applying the functor H_P, i.e., we have the induced exact sequence $0 \to H_P L \to H_P M \to H_P N \to 0$. Since $H_P N \in \mathrm{Ker}\, T_P^{i \geqslant 1}$, we obtain $L \in \mathrm{Stat}(P)$ as in the proof of Proposition 3.6.

Now suppose that $L \in \mathrm{Stat}(P)$. By applying the functor H_P, we obtain an induced exact sequences $0 \to H_P L \to H_P M \to X \to 0$ and $0 \to X \to^f H_P N \to Y \to 0$ for some

X, Y. After applying the functor T_P to the first sequence, we then have the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^A(P,X) \arrow[r] & T_P H_P L \arrow[r] \arrow[d, "\rho_L"] & T_P H_P M \arrow[r] \arrow[d, "\rho_M"] & T_P X \arrow[r] \arrow[d, "h"] & 0 \\
 & 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Note that $H_P L, H_P M \in \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$, so we have that $\mathrm{Tor}_i^A(P,X) = 0$ for all $i \geqslant 2$ by dimension shifting. Moreover, ρ_L and ρ_M are isomorphisms, since $L, M \in \mathrm{Stat}(P)$. It follows from the last diagram that $\mathrm{Tor}_1^A(P,X) = 0$ and $h = (T_P f)\rho_N$ is an isomorphism. Now we have that $X \in \mathrm{Ker}\, T_P^{i\geqslant 1}$ and that $T_P f$ is an isomorphism since ρ_N is an isomorphism. Note also that, by applying the functor T_P to the sequence $0 \to X \to^f H_P N \to Y \to 0$, we have an induced exact sequence

$$0 \to \mathrm{Tor}_1^A(P,Y) \to T_P X \to^{T_P f} T_P H_P N \to T_P Y \to 0$$

and that $\mathrm{Tor}_i^A(P,Y) = 0$ for all $i \geqslant 2$, since $H_P N \in \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$. Hence we obtain that

$$\mathrm{Tor}_1^A(P,Y) = 0 = T_P Y$$

by arguments above. It follows that $Y \in \mathrm{Ker}\, T_P^{i\geqslant 0}$. Then $Y = 0$ by assumptions and hence $X \simeq H_P N$ canonically. Therefore, we deduce that the functor H_P preserves the exactness of the exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{Stat}(P)$.

Finally, we conclude that P is a $*^s$-module. □

In fact, the assumptions in the last result also characterize $*^s$-modules. To see this, we need the following lemma.

Lemma 4.2. *Let P be a selfsmall R-module with $A = \mathrm{End}_R P$. Then $\Omega^2(\mathrm{Ker}\, T_P^{i\geqslant 1}) \subseteq \mathrm{Costat}(P)$, where Ω^2 denotes the second syzygy module.*

Proof. For any $N \in \Omega^2(\mathrm{Ker}\, T_P^{i\geqslant 1})$, we have an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_1, A_2 \in {}_A\mathcal{P}$ and $M \in \mathrm{Ker}\, T_P^{i\geqslant 1}$. Then we have an induced exact sequence $0 \to T_P N \to T_P A_2 \to T_P A_1 \to T_P M \to 0$ by applying the functor T_P. Now after applying the functor H_P to the last sequence, we obtain an induced exact sequence $0 \to H_P T_P N \to H_P T_P A_2 \to H_P T_P A_1$. Note that σ_{A_2} and σ_{A_1} are isomorphisms since P is selfsmall, so we deduce that σ_N is also an isomorphism. Hence $N \in \mathrm{Costat}(P)$. □

Let P be a $*^s$-module. We have seen that ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$ in Proposition 2.3. Now we will show that $\mathrm{Ker}\, T_P^{i\geqslant 0} = 0$ in this case.

Proposition 4.3. *Let P be a $*^s$-module. Then $\mathrm{Ker}\, T_P^{i\geqslant 0} = 0$.*

Proof. For any $M \in \operatorname{Ker} T_P^{i \geqslant 0}$, we have an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_1, A_2 \in {}_A\mathcal{P}$. Then, by applying the functor T_P to the sequence, we obtain an induced exact sequence $0 \to T_P N \to T_P A_2 \to T_P A_1 \to 0$. Note that $T_P N \in \operatorname{Stat}(P)$ since $N \in \operatorname{Costat}(P)$ by Lemma 4.2, so, after applying the functor H_P, we obtain that the induced sequence $0 \to H_P T_P N \to H_P T_P A_2 \to H_P T_P A_1 \to 0$ is exact by Proposition 2.2. It follows that $M = \operatorname{Coker}(A_2 \to A_1) \simeq \operatorname{Coker}(H_P T_P A_2 \to H_P T_P A_1) = 0$. □

Combining Propositions 2.3, 4.1 and 4.3, we obtain the following characterization of $*^s$-modules.

Theorem 4.4. *Let P be an R-module with $A = \operatorname{End}_R P$. Then P is a $*^s$-module if and only if ${}_A\mathcal{P} \subseteq \operatorname{Costat}(P) \subseteq \operatorname{Ker} T_P^{i \geqslant 1}$ and $\operatorname{Ker} T_P^{i \geqslant 0} = 0$.*

Specially, we have the following result.

Proposition 4.5. *Let P be an R-module with $A = \operatorname{End}_R P$. Assume that $\operatorname{Costat}(P) = \operatorname{Ker} T_P^{i \geqslant 1}$, then P is a $*^s$-module.*

Proof. Under assumptions we clearly have that ${}_A\mathcal{P} \subseteq \operatorname{Costat}(P)$. Moreover, for any $M \in \operatorname{Ker} T_P^{i \geqslant 0}$, we obtain that $M \in \operatorname{Costat}(P) = \operatorname{Ker} T_P^{i \geqslant 1}$. Hence $M \simeq H_P T_P M = 0$. By Theorem 4.4, we see that P is a $*^s$-module. □

We can now give a characterization of $*^s$-modules which is similar to Theorem 1.1(3).

Proposition 4.6. *Let P be a selfsmall R-module. The following are equivalent.*

(1) *P is a $*^s$-module.*
(2) *For any exact sequence $0 \to L \to P_M \to M \to 0$ with $P_M \in \operatorname{Add} P$ and $M \in \operatorname{Stat}(P)$, $L \in \operatorname{Stat}(P)$ if and only if the functor H_P preserves the exactness of the sequence.*

Proof. (1) ⇒ (2) is followed from the definition of $*^s$-module.

(2) ⇒ (1). As proved in Proposition 2.2(2), we can obtain that, for any $M \in \operatorname{Stat}(P)$, there is an infinite exact sequence $\cdots \to P_n \to \cdots \to P_1 \to M \to 0$ which remains exact after applying the functor H_P, where $P_i \in \operatorname{Add} P$ for each i. Consequently, we have that ${}_A\mathcal{P} \subseteq \operatorname{Costat}(P) \subseteq \operatorname{Ker} T_P^{i \geqslant 1}$ as in the proof of Proposition 2.3(1). Finally, the same proof as in Proposition 4.3 yields that $\operatorname{Ker} T_P^{i \geqslant 0} = 0$. Thus we obtain that P is a $*^s$-module by Theorem 4.4. □

5. $*^s$-Modules with special properties

This section is devoted to the study of $*^s$-modules P such that the subcategory $\operatorname{Stat}(P)$ has some special properties.

We have seen that selfsmall R-modules which are faithfully flat over their endomorphism rings are $*^s$-modules before. The following result shows that they are just $*^s$-modules which are flat over their endomorphism rings.

Proposition 5.1. *Let P be a selfsmall R-module with $A = \mathrm{End}_R P$. The following are equivalent.*

(1) *P_A is faithfully flat.*
(2) *P is a $*^s$-module and P_A is flat.*
(3) *P is a $*^s$-module and* $\mathrm{Stat}(P)$ *is closed under kernels of homomorphisms.*

Proof. (1) $\Rightarrow$ (2). By Proposition 3.6.

(2) $\Leftrightarrow$ (3). It is well known that P_A is flat if and only if $\mathrm{Stat}(P)$ is closed under kernels of homomorphisms (see for instance [19]).

(2) $\Rightarrow$ (1). Since P_A is flat and P is a $*^s$-module, we have that $\mathrm{Ker}\, T_P = \mathrm{Ker}\, T_P^{i\geqslant 0} = 0$ by Proposition 4.3. Hence P_A is faithfully flat [3]. □

For a $*^s$-module P, we know that $\mathrm{Costat}(P)$ is a resolving subcategory from Proposition 2.3(2). Dually, we can consider when $\mathrm{Stat}(P)$ is a coresolving subcategory. The following gives some characterizations of this case. We note that the class of such $*^s$-modules contains all tilting modules of projective dimension $\leqslant 1$.

Theorem 5.2. *Let P be an R-module and $\mathcal{I}$ be the class of all injective R-modules. The following are equivalent.*

(1) *P is a $*^s$-module such that* $\mathrm{Stat}(P)$ *is a coresolving subcategory.*
(2) *P is selfsmall and* $\mathrm{Stat}(P) = P^{\perp}$.
(3) *P is selfsmall and* $\mathrm{Stat}(P) \subseteq P^{\perp} \subseteq \mathrm{Gen}(P)$.
(4) $\mathcal{I} \subseteq \mathrm{Stat}(P) \subseteq P^{\perp}$ *and* ${}_A\mathcal{P} \subseteq \mathrm{Costat}(P) \subseteq \mathrm{Ker}\, T_P^{i\geqslant 1}$.

Proof. (1) $\Rightarrow$ (2). Let $S \in \mathrm{Stat}(P)$ and $0 \to S \to I_1 \to S_1 \to 0$ be an exact sequence with $I_1 \in \mathcal{I}$. Since $\mathrm{Stat}(P)$ is a coresolving subcategory, $\mathcal{I} \subseteq \mathrm{Stat}(P)$ and $\mathrm{Stat}(P)$ is closed under cokernels of monomorphisms. Hence we have that $S_1 \in \mathrm{Stat}(P)$. It follows from Proposition 2.2 that the sequence remains exact after applying the functor H_P. Then we obtain that $\mathrm{Ext}_R^1(P, S) = 0$ for any $S \in \mathrm{Stat}(P)$. Therefore, $\mathrm{Stat}(P) \subseteq P^{\perp}$ since $\mathrm{Stat}(P)$ is a coresolving subcategory.

On the other hand, we may consider the exact sequence $0 \to S \to I_1 \to I_2 \to X \to 0$ with $I_1, I_2 \in \mathcal{I}$, for any $S \in P^{\perp}$. It clearly remains exact after applying the functor H_P. Hence we have an induced exact sequence $0 \to H_P S \to H_P I_1 \to H_P I_2 \to H_P X \to 0$. By applying the functor T_P to the sequence, we obtain that $X = \mathrm{Coker}(I_1 \to I_2) \simeq \mathrm{Coker}(T_P H_P I_1 \to T_P H_P I_2) = T_P H_P X$ canonically since $\mathcal{I} \subseteq \mathrm{Stat}(P)$. That is, $X \in \mathrm{Stat}(P)$. Since P is a $*^s$-module, we see that $\mathrm{Im}(I_1 \to I_2) \in \mathrm{Stat}(P)$ and similarly $S \in \mathrm{Stat}(P)$. Hence we have also $P^{\perp} \subseteq \mathrm{Stat}(P)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4). By assumptions we have that $\operatorname{Add} P \subseteq P^{\perp}$. Now for any $M \in P^{\perp}$, we have an exact sequence $0 \to M_1 \to P_1 \to^{f_1} M \to 0$ with $P_1 = P^{(H_P M)}$ and f_1 the evaluation map, since $P^{\perp} \subseteq \operatorname{Gen}(P)$. Note the sequence clearly stays exact after applying the functor H_P and since $P_1, M \in P^{\perp}$, we obtain that $M_1 \in P^{\perp}$ too. By repeating the process to M_1, and so on, we finally obtain an infinite exact sequence $\cdots \to P_n \to \cdots P_1 \to M \to 0$, with $P_i \in \operatorname{Add} P$ for each i, such that the sequence remains exact after applying the functor H_P. It follows that $M \in \operatorname{Stat}(P)$ by [19]. Hence we deduce that $\operatorname{Stat}(P) = P^{\perp}$. Moreover, by an argument similar to the proof of Proposition 2.3(1), we obtain that ${}_A\mathcal{P} \subseteq \operatorname{Costat}(P) \subseteq \operatorname{Ker} T_P^{i\geqslant 1}$ since P is also selfsmall.

(4) $\Rightarrow$ (1). Since $\mathcal{I} \subseteq \operatorname{Stat}(P) \subseteq P^{\perp}$, we easily check that $\operatorname{Stat}(P)$ is a coresolving subcategory by a dual proof of Proposition 2.3(2). Now it remains to show that P is a $*^s$-module. By Theorem 4.4, we need only to prove that $\operatorname{Ker} T_P^{i\geqslant 0} = 0$. This will be proved similarly as in Proposition 4.3. Namely, let $M \in \operatorname{Ker} T_P^{i\geqslant 0}$ and take an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_2, A_1 \in {}_A\mathcal{P}$. Then we have an induced exact sequence $0 \to T_P N \to T_P A_2 \to T_P A_1 \to 0$ by applying the functor T_P. Note that $N \in \operatorname{Costat}(P)$ by Lemma 4.2, so that $T_P N \in \operatorname{Stat}(P) \subseteq P^{\perp}$. It follows that there is an induced exact sequence $0 \to H_P T_P X \to H_P T_P A_2 \to H_P T_P A_1 \to 0$ by applying the functor H_P. Hence we obtain that $M = \operatorname{Coker}(A_2 \to A_1) \simeq \operatorname{Coker}(H_P T_P A_2 \to H_P T_P A_1) = 0$. $\square$

In particular, if P is a small $*^s$-module and $\operatorname{Stat}(P)$ is coresolving and closed under homomorphic images, then $P^{\perp}$ is a 1-tilting class.

Clearly, for P a $*^s$-module, $\operatorname{Stat}(P)$ cannot be resolving except that P is a progenerator. However, it may happen that $\operatorname{Stat}(P)$ is closed under kernels of epimorphisms or extensions.

Recall that an R-module P is Σ-direct-projective if any epimorphism $f : L \to N$ with $L, N \in \operatorname{Add} P$ splits (see for instance [19]).

Proposition 5.3. *Let P be a $*^s$-module. The following are equivalent.*

(1) $\operatorname{Stat}(P)$ *is closed under kernels of epimorphisms.*
(2) *P is Σ-direct-projective.*
(3) $\operatorname{Ker} T_P = 0$.

Proof. (1) $\Rightarrow$ (2). For any epimorphism $f : P_1 \to P_0$ with $P_0, P_1 \in \operatorname{Add} P$, let L be the kernel of f, then $L \in \operatorname{Stat}(P)$ by assumption. Since P is a $*^s$-module, we have an induced exact sequence $0 \to H_P L \to H_P P_1 \to H_P P_0 \to 0$ after applying the functor H_P by Proposition 2.2(1). Note that the sequence splits since $H_P P_0$ is projective, so we obtain that $f \simeq T_P H_P f$ splits too. It follows that P is Σ-direct-projective.

(2) $\Rightarrow$ (3). Let $M \in \operatorname{Ker} T_P$ and take an exact sequence $A_1 \to^{f} A_0 \to M \to 0$ with $A_0, A_1 \in {}_A\mathcal{P}$. Then we have an induced exact sequence $T_P A_1 \to^{T_P f} T_P A_0 \to 0$ by applying the functor T_P. Since P is Σ-direct-projective, $T_P f$ splits. It follows that the induced sequence $H_P T_P A_1 \to^{H_P T_P f} H_P T_P A_0 \to 0$ is exact. Hence we have that $M = \operatorname{Coker} f \simeq \operatorname{Coker} H_P T_P f = 0$.

(3) $\Rightarrow$ (1). For any exact sequence

$$0 \to L \to M \to N \to 0 \tag{$*$}$$

with $M, N \in \mathrm{Stat}(P)$, we have an induced exact sequence $0 \to H_P L \to H_P M \to H_P N \to X \to 0$ by applying the functor H_P. By applying the functor T_P to the last sequence we also have an induced exact sequence $T_P H_P M \to T_P H_P N \to T_P X \to 0$. Then, $T_P X = \mathrm{Coker}(T_P H_P M \to T_P H_P N) \simeq \mathrm{Coker}(M \to N) = 0$. It follows that $X = 0$ by assumptions. Hence the exact sequence $(*)$ remains exact after applying the functor H_P. Since P is a $*^s$-module, we conclude that $L \in \mathrm{Stat}(P)$ and $\mathrm{Stat}(P)$ is closed under kernels of epimorphisms. $\square$

As a corollary, we have the following result.

Corollary 5.4.

(1) *P is a quasi-progenerator if and only if P is a Σ-direct-projective $*$-module.*
(2) *P is a 2-quasi-progenerator if and only if P is a Σ-direct-projective $*^2$-module.*

Proof. (1) The necessary part is obvious. As to the sufficient part, note that $\mathrm{Gen}(P) = \mathrm{Stat}(P)$ since P is a $*$-module, so we have that $L \in \mathrm{Gen}(P)$, for any exact sequence $0 \to L \to P' \to N \to 0$ with $P' \in \mathrm{Add}\, P$, by assumptions and Proposition 5.3. In another word, P is a Σ-self-generator. Now the conclusion follows from Proposition 3.5.

(2) Similarly, we need only to show that, if P is a Σ-direct-projective $*^2$-module, then P is a 2-quasi-progenerator. Note that $\mathrm{Pres}(P) = \mathrm{Stat}(P)$ since P is a $*^2$-module. Now consider any exact sequence $0 \to L \to P_2 \to P_1 \to N \to 0$ with $P_1, P_2 \in \mathrm{Add}\, P$ and let $M = \mathrm{Ker}(P_1 \to N)$. By Proposition 5.3, we obtain that $M \in \mathrm{Pres}(P)$, and consequently $L \in \mathrm{Pres}(P)$. It follows that the sequence remains exact after applying the functor H_P by Proposition 2.2(1). Hence P is $(2, 2)$-quasi-projective and, consequently, is a 2-quasi-progenerator by [17]. $\square$

We end with the following result, which deals with when the subcategory $\mathrm{Stat}(P)$ is closed under extensions. The proof is similar to that of the corresponding result on $*$-modules [5], so we omit it here.

Proposition 5.5. *Let P be a $*^s$-module. Then $\mathrm{Stat}(P)$ is closed under extensions if and only if $\mathrm{Ext}^1_R(P, M) = 0$ for any $M \in \mathrm{Stat}(P)$.*

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